THE LIFE AND SERVICES OF MAXIME BÔCHER.*

MAXIME BÔCHER was born in Boston, August 28, 1867, and died at his home in Cambridge, September 12, 1918. His father, Ferdinand Bôcher, was the first professor of modern languages at the Massachusetts Institute of Technology. Shortly after Mr. Charles W. Eliot, at that time professor of analytical chemistry and metallurgy in the same institution, became President of Harvard University, Professor Bôcher was called to Cambridge (in 1872) and for three decades was one of the leading teachers in the faculty of Harvard College. He was an enthusiastic collector of books. His library, which was divided after his death, formed the nucleus of the library of the French Department and yielded, furthermore, a welcome accession to the library of the Cercle Français; but its most important part, the valuable Molière and Montaigne collections, passed intact to the library of Harvard College. It was through the generosity of Mr. James Hazen Hyde, who bought the whole library, that such a disposition of the books became possible.

As a college teacher Ferdinand Bôcher is remembered by many men for whom college life in their student days offered varied attractions, as one who helped them to see and enjoy the beauty of language and literature.

Maxime's mother was Caroline Little, of Boston. She was of Pilgrim ancestry, being a descendant of Thomas Little, who joined the Plymouth Colony in its early days and in 1633 married Anne Warren, the daughter of Richard Warren, who came in the Mayflower.

Thus Bôcher's boyhood was passed in a home in which much that is best in the spirit and thought of France was united with the traditions and intellectual life of New England. He attended various schools, both public and private, in Boston and Cambridge; but it was to the influence of his parents that the awakening of his interest in science was due.

He graduated at the Cambridge Latin School in 1883 and took the bachelor's degree at Harvard in 1888. Then followed three years of study at Göttingen, where he received the degree

* For a critical estimate of Bôcher's scientific work the reader is referred to Professor Birkhoff's article in the February BULLETIN.

of doctor of philosophy in 1891, and at the same time the prize offered in mathematics by the philosophical faculty of the university. From 1891 till his death he was a member of the Department of Mathematics in Harvard University. He married Miss Marie Niemann, of Göttingen, in 1891. His wife and three children, Helen, Esther, and Frederick, survive him.

His college course was a broad one. Outside of his main field of mathematics and the neighboring field of physics he took a course in Latin and two courses in chemistry, and courses in philosophy under Professor Palmer, in zoology under Professor Mark, and in physical geography and meteorology under Professor Davis; and it is interesting to note that in his senior year, beside his work in mathematics, he elected Professor Norton's course in Roman and mediæval art; a course in music with Professor Paine, and an advanced course in geology with Professor Shaler and Professor Davis, and Professor (then Mr.) Wolff. In his senior year he also competed for a Bowdoin Prize, and the committee awarded him a second prize for an essay on "The meteorological labors of Dove, Redfield, and Espy." At graduation he received the bachelor's degree summa cum laude, with highest honors in mathematics, his thesis being "On three systems of parabolic coordinates." A travelling fellowship was granted him, and it was twice renewed.

Bôcher's education was not confined to the courses he took. He was a reader and a thinker, and he was interested in many of the general questions of the day. But generalities did not satisfy him; he demanded of himself that he know precisely the essential facts. His critical powers were early cultivated, and he was endowed with good judgment. In debate, he was able to marshall his facts with rapidity, to arrange them strategically, and to make his point with clearness. In rebuttal, he was an expert.

I recall an incident which occurred at a meeting of the M. P. Club* in the early nineties, and which shows the characteristics last mentioned. Professor Woods had given an interesting talk on surfaces which are applicable to one another, and had illustrated his subject with models from the Brill collection in the mathematical library of the Institute.

* This club was formed in the eighties for the purpose of bringing the members of the departments of mathematics and physics at the Massachusetts Institute of Technology and Harvard into closer relations.

One of the members of the Club was a physician, whose interest in mathematics had been kindled by Benjamin Peirce, and who, though not a profound mathematician, nevertheless delighted to read mathematics, much as our ancestors read their Horace. He asked a question which was based on his doubt whether parallel lines, in any logically necessary interpretation of the words, "meet at infinity." Now, there was also present a learned professor from another institution, and it pleased him to answer the doctor from a mighty height. But, in his answer, he was thinking only of projective geometry, and his arrogance made Bôcher indignant. "That is not true in the geometry of inversion," the latter replied. "That is not geometry," was the professor's scornful rejoinder. "It is what Klein calls geometry," came back quick as a flash. "Oh, Klein is not a geometer." This was the professor's last shot. In two brief statements of facts the youthful Bôcher had put his opponent into the position of asserting that the man who wrote the "Vergleichende Betrachtungen über neuere geometrische Forschungen" was not a geometer!

Above all, Bôcher was sincere. He liked to argue and to defend a position; but when the game was over, it was the truth which had been brought out that pleased him most.

He distrusted popular conclusions, even when the public was a learned one. It was facts, not views, that he sought, and his own intellect was the final arbiter. The following incident is characteristic of his type of mind. When his last sickness was developing, he needed a physician, and the well-known doctors were away in the war. He made inquiries one day regarding a young practitioner of rising fame, with whom Professor Birkhoff had recently had some experience. The latter said in closing, "I must add, however, that Dr. —— is pessimistic. He is given to taking a gloomy view of the condition of his patients." "I do not care whether he is pessimistic or not," was Bôcher's reply, "if the diagnosis is correct."

The later years of his life were not happy ones. Even as far back as the winter of 1913–14 his strength was frequently inadequate for the daily needs. He never complained; in fact, he was unwilling to talk about himself even for a moment. But for one whose demands on himself were such as Bôcher's it must have been a severe trial not to achieve the full measure of results of which the mind was capable and for which it longed to work.

He was a Puritan, and with the virtues he had also the faults of the Puritan. There was no place in his world for human weakness, even though the individual had done his best. A reverence for human beings because of their struggles to attain higher things was lacking in his make-up; he respected only results. And so, to many a man who came into personal relations with him in his profession, he seemed cold and unsympathetic. What the stranger, however, too often failed to observe was that Bôcher applied the same stern standards to himself. Why should others expect to fare better?

In order to understand the mathematical work of Bôcher it is well to consider at the outset the state of the science as he found it. The nineteenth century was an era of intense mathematical activity, not in one land alone, but among all the peoples which were leaders in scientific thought. If it was not reserved for mathematicians to make formal discoveries coordinate in importance with those which formed the crown of the discoverers and early developers of the calculus, it is none the less true that mathematical imagination never played more freely, not only in geometry and algebra, but also in analysis and mathematical physics.

But mathematics was no longer in its infancy. In the great age just preceding the French revolution, a mathematician could know, at least in its essential parts, all that had been done in the science up to that time, just as, a century earlier, the man of learning was conversant not only with mathematics and physics, but also with the principal systems of philosophy. With the enormous expansion of the subject matter, or detailed theories, which grew up and flourished with amazing virility in an age characterized by its struggle for intellectual freedom, a point had been reached where it seemed as if mathematics was destined to disintegrate through the very volume of its scientific content.

It was at this time—the eve of the Franco-Prussian War—that two youths met in Berlin, who were to become leaders in mathematical thought—Felix Klein and Sophus Lie. True, these men were remote from each other in their specific mathematical interests, and their life work lay in different fields. Lie built up a consistent, harmonious theory which both yielded new results and brought old ones under a common point of view. With Klein it was not a question of developing

a method for its own sake, or even of caring for method, except in so far as he was thus able to uncover the natural interrelations of parts of the science which hitherto had seemed foreign to each other.

A pupil of Clebsch and Plücker, Klein early became acquainted with the geometric advances that group themselves about the names of Monge and Poncelet, of Steiner and von Staudt. In analysis, the theory of functions, as developed by Cauchy and his followers, was already beginning to come into its own. Göttingen was filled with the traditions of Riemann, whose life touched fingers with that of Klein. In algebra, Galois's contributions were still new, and Hermite and Kronecker had, hardly more than a decade previously, solved the general equation of the fifth degree.

Klein's first great contribution toward unifying apparently unrelated disciplines was the Erlanger Programm of 1872 mentioned above, on a Comparative Consideration of Recent Advances in Geometry. It was here that he set forth projective geometry, not as an isolated science—geometry, par excellence—but rather as one (true, the most important) of a whole array of geometries, of which, in particular, the geometry of reciprocal radii, or inversion, is a member; for the basis of each of the geometries is the group of transformations which leave invariant certain configurations, and two geometries are essentially equivalent when their groups are isomorphic and their elements stand to each other in a one-to-one and continuous relation.

It would have been easy for Klein at this stage to found a school on the basis of postulates. If the thought ever occurred to him, he rejected it both because the results to be obtained would lack important mathematical content and because he instinctively sought the specific interrelations of seemingly distinct branches of mathematics, in order that one might yield new theorems, or illumine old ones, in the other.

It was to an environment imbued with such traditions that Bôcher came, when he was matriculated at Göttingen in the fall of 1888. His previous training at Harvard had prepared him to enter at once on advanced work. In the last year of his college course, as has already been said, he had written a thesis on parabolic coordinates. Klein was beginning the continuation of his lectures on the potential function,

and these were followed by his lectures on the partial differential equations of mathematical physics, and on the functions of Lamé. He also lectured at this time on non-euclidean geometry.

It is seldom that a student is brought into such vital contact with the chief branches of mathematics as was the case with Bôcher. His thesis was on Developments of the Potential Function into Series, a subject which he shortly after worked out at greater length in a monograph. Though the leading ideas had been set forth by Klein in his lectures, nothing could be further from the truth than to think that Bôcher merely elaborated some details. The subject was an exceedingly broad one. It required for its treatment not so much a specific knowledge of the theory of the potential, although Bôcher was thoroughly equipped on that side; nor even familiarity with the geometry of inversion, of which he had made himself master; but rather, the power to carry through a piece of detailed analytic investigation with accuracy and skill, and with this work Klein occupied himself only in the most general way. Nor was it a question of the proof of convergence for the series obtained. Indeed, these proofs have not as yet been given, though recent advances have been made by Hilbert with the aid of integral equations. The importance of the dissertation in its influence on Bôcher lay largely in the fact that it stimulated his interest in mathematical physics, in pure geometry, in algebra, and in applied analysis. More precisely, beside the general geometrical ideas and theories above mentioned, the specific study of the Dupin cyclides and their generalization by Laguerre, Moutard, and Darboux was involved.* Through the method of elementary divisors, he was led to examine in detail a chapter in pure algebra, together with its application in more than a single field in geometry. From the formal solution of the first boundary value problem for Laplace's equation by means of series to the study of boundary value problems for the partial differential equations of physics of other than the elliptic type and the treatment of these problems by the more recently developed methods of integral equations, was a natural course. Throughout all his work, the total linear homogeneous differential equations of the second order were a constant source of further investiga-

* An appreciation of Bôcher's contributions in this field will be found in Professor Birkhoff's article.

tions, both by himself and by his pupils, and his last great published work, the Paris lectures, is in this field.

In the fall of 1891 Bôcher began his career as university teacher, being appointed to an instructorship in the department of mathematics of Harvard University. It is the practice of that department to give to each of its members an elementary, an intermediate, and an advanced course. Bôcher's teaching, both elementary and advanced, was successful from the beginning. He did not have to "learn to teach"; teaching came to him naturally. Doubtless he was aided in this direction by the example of his father and the family traditions, for his mother had also been a teacher; nor were his parents the only ones of the immediate family who had been engaged in that profession. The standards of clearness, both in thought and expression, which characterize French men of letters and science, Bôcher made his own, not by a conscious effort, but through an inner driving force which made it a part of his very nature to find suitable expression for his ideas. "He never tried to be clear," Major Julian Coolidge wrote me this fall, "because his constitution was such that he did not know how to express his thoughts in any but the clearest form." I would not, however, be understood as saying that he achieved his success as a teacher without effort. He gave careful thought to the preparation of all his instruction, both as regards the choice of material and the presentation; but he was able to do this without serious loss of time or energy.

His intermediate course in the first year of his teaching was on modern geometry. Professor Byerly had already developed this course to a point which gave it an important place in the undergraduate instruction. The outlook on geometry which Bôcher had acquired under Klein enabled him to make the course still more effective as an introduction to the ideas and methods of the higher geometry of the present day. He gave this course repeatedly (about every other year) during the whole period of his service, and he was engaged in the preparation of his lectures for publication at the time of his last illness.*

* Since this course meant so much to Professor Bôcher, the reader will be interested in his description of it in the Departmental Pamphlet:

"3. INTRODUCTION TO MODERN GEOMETRY AND MODERN ALGEBRA.

The subjects considered in this course are:

(*a*) Affine Transformations; The use of Imaginaries in Geometry;

In the minds of some readers the word *lecture* in connection with a sophomore course may cause doubts as to the efficiency of the instruction. The objection is raised that sophomores cannot take notes and get only vague outlines of ideas which they cannot develop further. It must be remembered, however, that this course is a free elective, and that it is chosen by men who have interest in mathematics and capacity for its pursuit. Moreover, frequent exercises are assigned (as a rule, daily) which range all the way from simple tests on the essentials to problems whose solution is possible only for students who really dominate the methods. These problems are corrected and returned to the student.

So much by way of apology. Let me now say, with all aggressiveness, that it was largely to the lecture method that both Professor Bôcher and I owed the awakening of our interest in mathematics when we were undergraduates in Harvard College. The instruction thus imparted stimulated thought, and the exercises assigned developed power—the power to obtain new results, even in the undergraduate stage. It was with exultation that we followed courses given by this method, in which our mathematical powers grew before our very eyes. In saying this, I am also stating Bôcher's views, for he repeatedly expressed himself on this subject in conversation.

Bôcher's advanced course in the first year of his professional life took the form of a seminary, the subject being curvilinear coordinates and functions defined by differential equations. A part of the instruction consisted of formal lectures on the latter topic, and he thus began, even at that early date, to treat topics in a field of analysis in which he was to become eminent.

In the eighties, a number of American students of mathematics from various colleges went abroad, chiefly to Germany, for further instruction and guidance in mathematics. When

Abridged Notation; Homogeneous Coordinates; Intersection and Contact of Conics; Envelopes; Reciprocal Polars; The Parametric Representation of Straight Lines and Conics; Cross-Ratio; Projection and Collineation; Inversion.

(*b*) Complex Quantities; The Elements of the Theory of Equations; Determinants; The Fundamental Conceptions in the Theory of Invariants.

The portion of the course devoted to the geometrical subjects (*a*) will be two or three times as extensive as that devoted to the algebraical subjects (*b*), and the relations between these two parts of the course will be emphasized."

they returned, some of them became university teachers and strove, so far as in them lay, to give to their students advantages like those to be found in Europe at a mathematical center. Bôcher was one of this latter group. With rare discernment for problems of importance, on which advanced students might work with a reasonable prospect of success, he gave himself unstintingly to the task of helping such students to carry through pieces of investigation and to put their results into good form. He did not foster work on the part of his students by artificial means—by high praise or an appeal to ambition. He felt that the student must be possessed of idealism and must, of his own nature, find satisfaction in scientific activity; otherwise, the writing of a doctor's thesis would represent only a forced growth. At times, it seemed to the beginner in research that he was unappreciative. But the student who had capacity for mathematical investigation and loved the science found an open ear and a ready response when he came with a contribution of real scientific merit, be that contribution in itself large or small.

The awakening in the science of mathematics in this country was followed at once by the springing up of the New York Mathematical Society, which shortly after became the American Mathematical Society. Of the latter Bôcher early became a member, and he took a keen interest in its affairs, contributing to its BULLETIN and participating in all its activities. He and Professor Pierpont were the speakers at the first Colloquium given by its members—at Buffalo, in 1896. When the establishment by the Society of a journal devoted to research was under discussion, it was through his insight that a way out of the difficulties which seemed insurmountable was found. Among the older members of the Society were those who saw in the establishment of such a publication an unfriendly act toward the *American Journal of Mathematics*. At a meeting of about a dozen mathematicians, held in New York in the fall of 1898 to discuss the question, this view was represented by the late Dr. McClintock. Bôcher asked him if he saw any objection to the Society's publishing its *Transactions*. To the surprise of all, there came a prompt answer in the negative. The difficulty was overcome. The Society might not establish the "Journal of the American Mathematical Society," but it might publish the "Transactions of the American Mathematical Society"!

Producers of mathematical research are, as a rule, not facile in their expression. When one has been engaged in the protracted study of a problem, the early difficulties and the underlying ideas become obscured through familiarity with the facts, and the writer produces a paper hard to read. The early editors of the *Transactions* labored, and not without success, to impress on the contributors the importance of making easily accessible to the reader the main results and methods of the paper, and of showing the relation of the investigation to previous work. It was here that Bôcher's power as a critic was of great service. But a critic, to be helpful in such work, must be constructive. How admirably Bôcher was adapted for this undertaking, could not be shown more strikingly than by the opening paragraphs of his Dissertation, which are a model of what an introduction to a scientific paper of wide scope should be. He was not a member of the first editorial boards, for at that time the *Annals of Mathematics* had just been taken over by Harvard University, and he was doing similar work for that journal. But from the start he was in close touch with the editors of the *Transactions*, and his views on general questions and specific papers were helpful to them. Later, he served for two terms (with the exception of one year, in which he was absent from the country) on the editorial board.

He was president of the Society from 1908 to 1910. For his presidential address he took as the subject: "The published and unpublished works of Charles Sturm on algebraic and differential equations." He delivered the address in Chicago. The meeting will live in the memories of all who were present, especially in those of the eastern colleagues, as a particularly delightful occasion.

Bôcher's judgment of men, too, was sound, and those who had occasion to discuss nominations or appointments with him felt that a decision which had his approval could be trusted.

The breadth of Bôcher's knowledge of mathematics was accompanied by a true sense of perspective. His estimate of the importance of an investigation was extraordinarily sound. In his own work, this quality of mind was both a help and a hindrance. It helped him to choose well the problems which he and his students were to study. It can fairly be said that Bôcher never occupied himself with an unimportant problem.

On the other hand, the enthusiasm just of doing things in mathematics—the joy of living, so to speak—gives to one's mental work a momentum which carries it over the obstacles of disappointment and discouragement, when one effort and another fail to yield results, and along with much which is valueless for others there come, now and then, contributions worthy of a lasting place in the science. I will not say that Bôcher was without such enthusiasm; but he did not show it in his intercourse with others. His nature was reserved. He would not talk on personal matters relating to himself, and this disinclination extended even to his scientific work.

He was, however, glad to discuss the work of others with them. He was quick to grasp the central idea and often could express it more clearly than its author. The early meetings of the Society were prized by those who attended them less for the formal papers presented than for the informal gatherings in the evening or about the breakfast table. It was here that the real mathematical discussions took place, and who of those who had the rare good fortune to be associated with that little group will ever forget what Bôcher was to us in those days? His special field was analysis; but so broad were his sympathies and his learning that he usually took a leading part in the discussions. His criticism was always helpful, often constructive, and freely given in the finest spirit.

We have mentioned the Presidential Address. At the St. Louis Congress, in 1904, he delivered an address on "The fundamental conceptions and methods of mathematics." He gave a lecture at the Fifth International Congress of Mathematicians, at Cambridge, England, in 1912, his subject being: "Boundary problems in one dimension." In 1913–14 he was exchange professor at Paris. His opening lecture was of a general nature and was entitled: "Charles Sturm et les mathématiques modernes."

It was not until late that Bôcher occupied himself with the writing of text-books. He had published some expository articles, chief among which were the pamphlet on "Regular points of linear differential equations of the second order," Harvard University Press, 1896; an article on "The theory of linear dependence," *Annals of Mathematics*, (2) 2 (1901) and an "Introduction to the theory of Fourier's series," ibid., (2) 7 (1906); three years later he wrote Tract 10 of the Cambridge (England) series, entitled: "An introduction to the study of integral equations."

The Algebra appeared in 1907. Hitherto, books on algebra in the English language had been of the Todhunter type, or they had followed the lead of Salmon, through whom "Higher Algebra" came to mean specifically the study of the algebraic invariants of a linear transformation. What the mathematician needed to know of linear dependence and the theory of linear equations, of polynomials (factorization, resultants, and discriminants), the reduction of one or of two quadratic forms to normal type (including, perhaps, the rudiments of elementary divisors) he had to pick up as best he could. In no one place were they treated systematically, and most of the treatments were inadequate for the present day needs of the science.

Bôcher filled this gap in a thoroughly satisfactory manner. The Algebra was received with appreciation, both in this country and abroad, and at the suggestion of Professor Study a German translation was prepared. How thoroughly the work had been done originally is seen from the fact that practically no revision was needed.

The Trigonometry (written jointly with Mr. Gaylord) and the Analytic Geometry are so widely and intimately known as to require no detailed comment. These books present elementary subjects in a form accessible for elementary students, and treat them with a degree of accuracy, elegance, and perspective seldom attained by writers of text-books.

I have spoken of Klein's efforts to *unify* mathematics. Bôcher's aim may be described by saying that he strove to *clarify* mathematics. To illustrate by a single, but important example, let me consider the theory of functions of a complex variable. In the early nineties there were two distinct schools, and neither sought to aid or to learn from the other. Cauchy based his theory on the calculus of residues, obtaining Taylor's theorem as a corollary. With Weierstrass and Méray power series formed the foundation. The integral was more pliable and better adapted to the needs of the subject. But the questions which the critics had raised regarding limits, and in particular the reversal of the order in a double limit, had not been settled in a satisfactory manner for integrals, and even for series they were ignored by the writers of the Cauchy school. On the other hand, Weierstrass restricted his infinite processes to differentiation and power series. His treatment was rigorous, but clumsy, and the whole theory took on a formidable aspect.

Riemann's methods were thought of less as forming an independent theory than as yielding an important mode of treatment for certain classes of functions; e. g., the algebraic functions and their integrals, and the functions defined by linear differential equations or their resolvents; notably, the *P*-function and the automorphic functions.

In 1893–94 Bôcher gave for the first time in his career the introductory course on the theory of functions of a complex variable, and in the same year he repeated his course on functions defined by differential equations, laying stress on the complex theory. The subjective effect is obvious. For him, it could not be a question of developing the general theory of functions as an end in itself. He was interested in the theory as a tool—as a means of investigating, for example, the functions defined by differential equations. But he was interested in improving the tool, in developing better machinery than had come down to us. He cared nothing for the schools. He sought the simplest method for solving each problem.

Of course, he was rigorous. But for him, rigor was not a strait-jacket. For him, rigor was not something superimposed on a proof, already satisfactory to a normal mind, by a certain cult of mathematicians. If a proof was not rigorous, it was not *clear*—it had not succeeded in analyzing completely the situation. Not that, with him, there was no place for intuition in mathematics. Quite the reverse. He recognized clearly that rigor is relative, depending on the domain of conceptions and the logical maturity of the student, and he was a master of diagnosis in determining what his students required or could receive, and what their minds must reject.

His contributions of the kind we have been considering were not confined to improving proofs already complete. He discovered gaps and filled them; as in the case of the theorem that a function which is harmonic in the neighborhood of a point, that point excepted, and becomes infinite there, must be of the form (when $n = 2$):

$$u = k \log r + \omega,$$

where ω is harmonic at the point, also.

How extensive and how useful this work of Bôcher's was will become evident to any one who will turn to the writer's Funktionentheorie, volume I, and look up the references under Bôcher's name in the index. And what he did in this field,

he did in others. His Algebra, for example, affords numberless instances in point.

In the early years of our professional lives we were in constant intercourse over such matters. Each of us was seeking to clarify and simplify his subject. Neither of us regarded the theory of functions of a real or of a complex variable as an end in itself, for each had his own ulterior uses for the theory—Bôcher, his differential equations, both complex and real. In fact, for each of us the theory of functions was *applied mathematics*, and in presenting its subject matter and its methods to our students, our aim was to show them great problems of analysis, of geometry, and of mathematical physics which can be solved by the aid of that theory.

Bôcher was quick to grasp the large ideas of the mathematics that unfolded itself before our eyes in those early years. His attitude toward mathematics helped me to have the courage of my convictions. The Funktionentheorie is largely Bôcher's work, less through the specific contributions cited on its pages than through the influence he had exerted prior to 1897—long before a line of the book had been written. We worked together, not as collaborators, but as those who hold the same ideals and try to attain them by the same methods. It was constructive work, and in such Bôcher was ever eager to engage.

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A THEOREM ON LINEAR POINT SETS.

BY DR. HENRY BLUMBERG.

(Read before the American Mathematical Society December 28, 1918.)

LET A be any given linear point set. We define the "relative exterior measure* of A in the interval I" as $m_e(A, I)/l$, where $m_e(A, I)$ represents the exterior measure (Lebesgue) of the subset of A in I, and l is the length of I. We then define the "relative exterior measure of A at the point x" as

* Cf. Denjoy, *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 130.

k, if

$$\lim_{l_n \to 0} \frac{m_e(A, I_n)}{l_n} = k$$

for every sequence $\{I_n\}$ of intervals enclosing x and having a length l_n that approaches 0 with increasing n. It is the purpose of this note to prove the

THEOREM. *The relative exterior measure of every linear point set exists and is equal to* 1 *at every one of its points, with the possible exception of those of a set of measure* 0.

Proof. Let M represent the subset of the given set A that consists of points where the relative exterior measure of A is not 1. Hence there exists, for every point x of M, an integer n_x such that for every $\epsilon > 0$ there is an interval enclosing x and of length $< \epsilon$ in which the relative exterior measure of A is $< 1 - 1/n_x$. For a given x, let n_x' be the smallest integer having the property just described. Designating by M_n the set of x's for which $n_x' = n$, we obtain the decomposition

$$M = M_2 + M_3 + \cdots + M_n + \cdots.$$

Our theorem will be established if we show that M_n is of measure zero.

Let I, of length l, be any interval of the linear continuum. According to the definition of M_n, every point x of M_n may be enclosed in an arbitrarily small interval J_x in which the relative exterior measure of M and a fortiori of M_n is $<1-1/n$. The intervals J_x may be so chosen that they lie entirely in an interval I' of length $< l + \delta$, where δ is any given positive number. According to a well known theorem, we may select from the J_x's a denumerable infinity $[J', J'', J''', \cdots]$ having the same interior points as all the J_x's. Let m_n be the measure of the portion of I' covered by the intervals $J', J'', \cdots, J^{(n)}$. We distinguish two possibilities:

(1) $$\lim_{n \to \infty} m_n \leqq \tfrac{2}{3} l$$

and

(2) $$\lim_{n \to \infty} m_n > \tfrac{2}{3} l.$$

We shall prove that in either case the exterior measure of the subset of M_n in I is $\leqq [1 - (1/3n)]l$. In the first case, this is

evident. In the second case, suppose $m_\nu > \frac{2}{3}l$. It is easily shown by elementary reasoning that we may extract from the sequence $J', J'', J''', \cdots, J^{(\nu)}$ a sequence $\bar{J}', \bar{J}'', \cdots, \bar{J}^{(\bar{\nu})}$ which cover the same portion of I' and no part more than twice. Since the relative exterior measure of M_n in every J is $< 1 - 1/n$, it follows that the interior measure of the set complementary to M_n is greater than $\bar{l}_\sigma/n$ in every $\bar{J}_\sigma$ (of length $\bar{l}_\sigma$); and since no part of the continuum is covered by more than two $\bar{J}$'s, we conclude that the total interior measure of this complementary set in the portion covered by the $\bar{J}_\sigma$ $[\sigma = 1, 2, \cdots \bar{\nu}]$, is $> \frac{1}{2} \cdot (1/n) \cdot \frac{2}{3}l = l/3n$. Therefore the exterior measure of the subset of M_n in I is $< l + \delta - (l/3n)$; and since δ may be chosen arbitrarily small, this exterior measure is $\leqq l - (l/3n)$.

Having thus proved that the relative exterior measure of M_n is $\leqq 1 - (1/3n)$ in every interval, we may now show that the measure of M_n is zero. To this end, we show that *a linear set S whose relative exterior measure is $< 1 - k$, $k > 0$, in every interval is necessarily of measure zero.* For let m = exterior measure of S. For every given positive ϵ we may then enclose S in a set of intervals I_n of total length $< m + \epsilon$. Furthermore, the subset of S in each I_n may be enclosed in a set of intervals of total length $< (1 - k)l_n$, where l_n = length of I_n; and therefore the entire set S, in a set of intervals of total length $< \Sigma_n(1 - k)l_n \leqq (1 - k)(m + \epsilon)$. Therefore

$$(1 - k)(m + \epsilon) > m, \qquad m < \frac{(1 - k)\epsilon}{k},$$

and accordingly $m = 0$. Our theorem is thus proved.

Let Z be the subset of A of zero measure at the points of which the relative exterior measure of A is not 1; and let $H = A - Z$ be the remaining set. Since A and H differ by a set of zero measure, the relative exterior measure of the one is the same at every point as that of the other. Therefore H has the relative measure 1 at every one of its points, and may be thought of as "homogeneous" as to exterior measure. We thus have

COROLLARY 1. *Every linear point set A may be represented as*

$$A = H + Z,$$

where H is a ("homogeneous") set having relative exterior measure 1 *at every one of its points, and Z is of measure zero.*

We obtain a particular case of our theorem if we assume A to be a measurable set. Exterior measure will then be replaced by measure, and relative exterior measure by "relative measure." We thus have

COROLLARY 2. *The relative measure of a measurable set is* 1 *at every one of its points except possibly at those of a set of measure zero.*

Corollary 2 is equivalent to a theorem of Lebesgue-Denjoy.* The present note, therefore, also gives a very simple proof of this important theorem.

So far the author has not succeeded in proving the theorem of this note for higher dimensions, although there seems to be little ground for doubting its validity in n-space.

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A GENERAL FORM OF GREEN'S THEOREM.

BY PROFESSOR P. J. DANIELL.

IN this paper a form of Green's theorem is considered which applies, on the one hand, to the boundary of any set E, measurable Borel, and relates, on the other hand, to potential functions which satisfy a general integral form of Poisson's equation,

$$\int_{B(E)} \frac{\partial V}{\partial n}\, ds = \int_E d\alpha(x, y),$$

where $\alpha(x, y)$ is some function of limited variation in (x, y). In particular it can be used in mathematical physics in problems in which mass (or electric charge) is not distributed continuously.

Let $V_1(x, y)$, $V_2(x, y)$ be two potential functions defined and

* Lebesgue, Leçons sur l'Intégration, pp. 123–124, and Denjoy, loc. cit., pp. 132–137. "Relative measure" is equivalent with Denjoy's "épaisseur." Lebesgue's considerations are indirect (as far as the theorem in question is concerned), being based on properties of integrals. Denjoy's proof is direct, but still comparatively involved and long.

"differentiable"* in the fundamental square $J(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$; let

$$\frac{\partial V_1}{\partial x} = v_1, \quad \frac{\partial V_1}{\partial y} = -u_1, \quad \frac{\partial V_2}{\partial x} = v_2, \quad \frac{\partial V_2}{\partial y} = -u_2$$

be summable with their squares in J.

Furthermore assume that u_1, u_2 satisfy R_1; v_1, v_2 satisfy R_2.

R_1. The total variation of $u(x, y)$ varying $y(0 \leqq y \leqq 1)$ is a function of x, finite nearly everywhere and summable in $(0 \leqq x \leqq 1)$.

R_2. The same as R_1, with v in place of u, and with the rôles of x and y interchanged.

In a previous paper† the author has shown that, under these restrictions, we may express

$$\int_{B(E)} \frac{\partial V_1}{\partial n} ds = \int_{B(E)} u_1 dx + v_1 dy$$

in the form

$$\int_E d\alpha_1.$$

In the present paper it is further proved that

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (u_1 u_2 + v_1 v_2) dx dy.$$

Consider a rectangle r $(a \leqq x \leqq b, c \leqq y \leqq d)$ contained in J. We have

$$\int_{B(r)} u_1 dx = \int_a^b [u_1(x, c) - u_1(x, d)] dx$$

$$= -\int_a^b dx \int_c^d d_y u_1(x, y).$$

In this $\int_c^d d_y u_1(x, y)$ may be regarded as a Stieltjes integral with y as variable, which, by R_1, exists for nearly all values of x and is summable in x. Then

$$-\int_a^b dx \int_c^d d_y u_1(x, y) = \int_r d\alpha'(x, y)$$

* De la Vallée Poussin, Cours d'Analyse, 3d ed., vol. 1, § 147.

† P. J. Daniell, this BULLETIN, Nov., 1918.

will be an absolutely additive function of rectangles r. Similarly

$$\int_{B(r)} v_1 dy = \int_c^d [v_1(b, y) - v_1(a, y)]dy$$

$$= \int_c^d dy \int_a^b d_x v_1(x, y)$$

$$= \int_r d\alpha''(x, y),$$

$$\int_{B(r)} u_1 dx + v_1 dy = \int_r d\alpha(x, y),$$

where $\alpha = \alpha' + \alpha''$.

For nearly all values of x, $(\partial V_2/\partial y) = -u$ is uniformly bounded with respect to y, by R_1. Then, for nearly all values of x, $V_2(x, y)$ is an absolutely continuous function of y and must be of limited variation in y.

By a theorem on integration by parts,*

$$-\int_c^d d_y(V_2 u_1) = -\int_c^d V_2 d_y u_1 - \int_c^d u_1 d_y V_2$$

$$= -\int_c^d V_2 d_y u_1 + \int_c^d u_1 u_2 dy,$$

for nearly all values of x.

Again since V_2 is "differentiable" at every point of J, it is continuous† and uniformly bounded.

If $\max | V_2 | = K$, the total variation of $V_2 u_1$, varying $y (0 \leqq y \leqq 1)$ will be not greater than

$$K \times \text{variation of } u_1 + \int_0^1 | u_1 u_2 | \, dy.$$

u_1, u_2 are summable with their squares in J, so that $u_1 u_2$ is also summable in J, or $\int_0^1 | u_1 u_2 | \, dy$ is a summable function of x where it exists (nearly everywhere). Then $V_2 u_1$ will also satisfy R_1, and similarly $V_2 v_1$ will satisfy R_2.

* P. J. Daniell, *Transactions Amer. Math. Society*, October, 1918.

† De la Vallée Poussin, ibid.

Combining the various facts, we then obtain

$$\int_{B(r)} V_2 u_1 dx = -\int_a^b dx \int_c^d d_y(V_2 u_1)$$

$$= -\int_a^b dx \int_c^d V_2 d_y u_1 + \int_a^b dx \int_c^d u_1 u_2 dy$$

$$= \int_r V_2 d\alpha' + \int_r u_1 u_2 dx dy.$$

The change from repeated to double integrals is legitimate, in the first integral (Stieltjes) because V_2 is uniformly continuous, in the second because $u_1 u_2$ is summable in J. Similarly

$$\int_{B(r)} V_2 v_1 dy = \int_r V_2 d\alpha'' + \int_r v_1 v_2 dx dy.$$

Then

$$\int_{B(r)} V_2(u_1 dx + v_1 dy) = \int_r V_2 d\alpha_1 + \int_r (u_1 u_2 + v_1 v_2) dx dy.$$

This is an equation in which all three expressions are absolutely additive functions of rectangles, and $V_2 u_1$, $V_2 v_1$ satisfy R_1, R_2 respectively; therefore for any set E, measurable Borel, contained in J,

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (u_1 u_2 + v_1 v_2) dx dy.$$

This was the theorem to be proved and we can rewrite it in the form

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} ds = \int_E V_2 d\alpha_1 + \int_E (\text{grad } V_1 \cdot \text{grad } V_2) dS.$$

Corollary 1. Interchanging V_1, V_2 and subtracting,

$$\int_{B(E)} \left(V_2 \frac{\partial V_1}{\partial n} - V_1 \frac{\partial V_2}{\partial n} \right) ds = \int_E V_2 d\alpha_1 - \int_E V_1 d\alpha_2.$$

Corollary 2. Instead of this make $V_1 = V_2 = V$.

$$\int_{B(E)} V \frac{\partial V}{\partial n} ds = \int_E V d\alpha + \int_E \text{grad}^2 V dS.$$

Three dimensions. The case of three dimensions is more valuable in applications, and the proof is exactly similar. We content ourselves with the mere statement. [Our previous notation is altered; u_1 takes the place of v_1, and v_1 of $-u_1$.] Let V_1, V_2 be two potential functions, defined and "differentiable" in the cube J $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1,\ 0 \leqq z \leqq 1)$; let the six partial derivatives be summable with their squares in J; and let $\partial V_1/\partial x$, $\partial V_2/\partial x$ satisfy R_1; $\partial V_1/\partial y$, $\partial V_2/\partial y$ satisfy R_2; $\partial V_1/\partial z$, $\partial V_2/\partial z$ satisfy R_3.

R_1. The total variation of u varying $x(0 \leqq x \leqq 1)$ is a function of (y, z) finite nearly everywhere and summable in $(0 \leqq y \leqq 1,\ 0 \leqq z \leqq 1)$.

R_2, R_3. The same as R_1 with v, w in place of u and with cyclical interchanges of x, y, z.

If the element of normal is drawn outwards,

$$\int_{B(r)} \frac{\partial V_1}{\partial n} dS = \int_r d\alpha_1$$

is an absolutely additive function of rectangular parallelopipeds r and α_1 (x, y, z) is a function of limited variation in J. Then we can define for any set E, in J, measurable Borel,

$$\int_{B(E)} \frac{\partial V_1}{\partial n} dS = \int_E d\alpha_1$$

and Green's theorem becomes

$$\int_{B(E)} V_2 \frac{\partial V_1}{\partial n} dS = \int_E V_2 d\alpha_1 + \int_E (\text{grad } V_1 \cdot \text{grad } V_2) d \text{ vol.}$$

The two corresponding corollaries follow immediately.

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ROTATING CYLINDERS AND RECTILINEAR VORTICES.

BY PROFESSOR H. BATEMAN.

§ 1. *Rectilinear Vortex and Rotating Cylinder in a Stream of Incompressible Fluid.*

We shall assume that the rotation of the cylinder produces a circulation around the cylinder which may be approximately represented by placing a rectilinear vortex along the axis of the cylinder, a device which was adopted by Lord Rayleigh* in his paper "On the irregular flight of a tennis ball." Let V be the velocity of the stream, $2\pi k$ the circulation around the cylinder and $2\pi c$ the strength of the rectilinear vortex at the side of the cylinder. Assuming that the axis of this vortex is parallel to the axis of the cylinder, the motion is two-dimensional and we may represent the velocity potential ϕ and stream line function ψ as follows:

$$\phi + i\psi = U\left(z + \frac{a^2}{z}\right) + ik \log z + ic \log \frac{z - z_0}{z - z_1},$$

where a is the radius of the cylinder, $z = x + iy$ is a complex variable specifying the position of a point relative to the axis of the cylinder, z_0 specifies the position of the vortex, and z_1 that of its image.

Let (u, v) be the component velocities of the fluid at (x, y), (u_0, v_0) the component velocities of the vortex; then differentiating the above equation with regard to z we find that

$$u - iv = U\left(1 - \frac{a^2}{z^2}\right) + \frac{ik}{z} + ic\left[\frac{1}{z - z_0} - \frac{1}{z - z_1}\right].$$

In calculating (u_0, v_0) we ignore the infinite velocity produced by the vortex itself, consequently

$$u_0 - iv_0 = U\left(1 - \frac{a^2}{z_0^2}\right) + \frac{ik}{z_0} - ic\,\frac{1}{z_0 - z_1}.$$

* *Mess. of Math.* (1878); Scientific Papers, vol. 1, p. 344. See also Lamb's Hydrodynamics, 4th ed., 1916, p. 77; Greenhill, *Mess. of Math.*, vol. 9 (1880), p. 113. Report on Gyroscopic Theory, Report of the Advisory Committee for Aeronautics, No. 146, p. 238.

Let us write

$$z_0 = r_0 e^{i\theta_0}, \qquad \zeta_0 = r_0 e^{-i\theta_0}, \qquad z_1 = \frac{a^2}{r_0} e^{+i\theta_0};$$

then since

$$\frac{d\zeta_0}{dt} = u_0 - iv_0,$$

we find that

$$\frac{dr_0}{dt} = U\left(1 - \frac{a^2}{r_0^2}\right)\cos\theta_0,$$

$$r_0\frac{d\theta_0}{dt} = -U\left(1 + \frac{a^2}{r_0^2}\right)\sin\theta_0 - \frac{k}{r_0} + \frac{cr_0}{r_0^2 - a^2}.$$

In order that the vortex may be stationary we must have either $U = 0$ or $\cos\theta_0 = 0$. The former case has been discussed by Greenhill* both for a stationary and moving vortex. In the latter case U, k, c, and r_0 must be connected by the equation

$$(1) \qquad \mp U\left(1 + \frac{a^2}{r_0^2}\right) - \frac{k}{r_0} + \frac{cr_0}{r_0^2 - a^2} = 0.$$

To study the stability of this stationary vortex let R, Θ denote small displacements from the stationary position; then

$$\frac{dR}{dt} = \mp U\Theta\left(1 - \frac{a^2}{r_0^2}\right),$$

$$r_0\frac{d\Theta}{dt} = \pm 2UR\frac{a^2}{r_0^3} + \frac{k}{r_0^2}R - cR\frac{r_0^2 + a^2}{(r_0^2 - a^2)^2}.$$

$$\therefore \frac{d^2R}{dt^2} = \mp U\left(1 - \frac{a^2}{r_0^2}\right)\left[\pm 2U\frac{a^2}{r_0^4} + \frac{k}{r_0^3} - \frac{c(r_0^2 + a^2)}{r_0(r_0^2 - a^2)^2}\right]R.$$

When the upper sign is taken [$\theta_0 = +(\pi/2)$], the vortex is in stable equilibrium if

$$(2) \qquad 2U\frac{a^2}{r_0^3} + \frac{k}{r_0^2} > \frac{c(r_0^2 + a^2)}{(r_0^2 - a^2)^2}.$$

The velocity at the surface of the cylinder is determined from the formula

* Encyclopedia Britannica. Article on "Hydrodynamics." The present case may possibly also have been discussed by Greenhill in a paper, *Quart. J. Math.*, vol. 15 (1878), which is at present inaccessible to me.

$$(3)\quad u^2+v^2=\left[2U\sin\theta+\frac{k}{a}+\frac{c}{a}\,\frac{a^2-r_0^2}{a^2-2ar_0\cos(\theta-\theta_0)+r_0^2}\right]^2.$$

The component forces (X, Y) on the cylinder are given by the formula

$$X-iY=\tfrac{1}{2}\rho\int_0^{2\pi}(u^2+v^2)ae^{-i\theta}d\theta+\rho\int_0^{2\pi}\frac{\partial\phi}{\partial t}ae^{-i\theta}d\theta.$$

Expanding u^2+v^2 by Fourier's theorem, we find that the terms involving $\cos\theta$ and $\sin\theta$ are

$$4U\frac{k}{a}\sin\theta-4U\frac{c}{a}\sin\theta-4\frac{kc}{ar_0}\cos(\theta-\theta_0)$$
$$+4U\frac{ca}{r_0^2}\sin(\theta-2\theta_0)+4\frac{c^2r_0\cos(\theta-\theta_0)}{a(r_0^2-a^2)};$$

hence the value of the first integral is

$$X-iY=-2\pi\rho i[Uk-c(u_0-iv_0)].$$

On the other hand we have for $r=a$

$$\frac{\partial\phi}{\partial t}+i\frac{\partial\psi}{\partial t}=-ic\frac{u_0+iv_0}{ae^{i\theta}-r_0e^{i\theta_0}}+ic\frac{u_1+iv_1}{ae^{i\theta}-(a^2/r_0)e^{i\theta_0}},$$

$$\int_0^{2\pi}\frac{ae^{i\theta}d\theta}{ae^{i\theta}-r_0e^{i\theta_0}}=0,\qquad \int_0^{2\pi}\frac{ae^{-i\theta}d\theta}{ae^{i\theta}-r_0e^{i\theta_0}}=-2\pi\frac{a^2}{r_0^2}e^{-2i\theta_0},$$

$$\int_0^{2\pi}\frac{ae^{i\theta}d\theta}{ae^{i\theta}-(a^2/r_0)e^{i\theta_0}}=2\pi,\qquad \int_0^{2\pi}\frac{ae^{-i\theta}d\theta}{ae^{i\theta}-(a^2/r_0)e^{i\theta_0}}=0,$$

$$\int_0^{2\pi}a(\dot\phi+i\dot\psi)\cos\theta d\theta=\pi ic\frac{a^2}{r_0^2}(u_0+iv_0)e^{-2i\theta_0}+\pi ic(u_1+iv_1),$$

$$\int_0^{2\pi}a(\dot\phi+i\dot\psi)\sin\theta d\theta=-\pi c\frac{a^2}{r_0^2}(u_0+iv_0)e^{-2i\theta_0}+\pi c(u_1+iv_1),$$

$$\int_0^{2\pi}\frac{\partial\phi}{\partial t}a\cos\theta d\theta=\pi c\frac{a^2}{r_0^2}(u_0\sin2\theta_0-v_0\cos2\theta_0)$$
$$-\pi cv_1=-2\pi cv_1,$$

$$\int_0^{2\pi}\frac{\partial\phi}{\partial t}a\sin\theta d\theta=-\pi c\frac{a^2}{r_0^2}(u_0\cos2\theta_0+v_0\sin2\theta_0)$$
$$+\pi cu_1=2\pi cu_1.$$

Hence we finally obtain the formulas

$$X = 2\pi\rho c(v_0 - v_1), \quad Y = 2\pi\rho(Uk - cu_0 + cu_1).$$

For a stationary vortex $X = 0$, $Y = 2\pi\rho Uk$ and the transverse force is the same as in Rayleigh's case when $c = 0$ and there is no vortex.*

For a stationary vortex (1) and (2) give the inequality

$$-U > \frac{2ca^2r_0^3}{(r_0^2 - a^2)^3};$$

hence if U is positive and $\theta_0 = \pi/2$, c must be negative and k negative. This means that the circulations around the vortex and cylinder are both in the counterclockwise direction and that the force $2\pi\rho Uk$ is negative. Hence this force tends to move the cylinder away from the region where the cylinder and stream of air are moving in opposite directions. This phenomenon attracted the attention of Newton in 1671 and was the subject of some experiments by Magnus in 1852. Many other experiments have been made since this time and in his beautiful lecture on† "The dynamics of a golf ball" Sir Joseph Thomson showed with the aid of a pressure gauge that there is indeed a difference of pressure on the two sides of a golf ball rotating in a stream of air and remarked that when a golf ball is in flight this difference in pressure may provide a lifting force greater than the weight of the ball, so that a rotating golf ball is a kind of flying machine. This

* In this case the formula gives Rayleigh's law that the transverse force is proportional to the velocity of the stream relative to the cylinder and the velocity of spin. In the case of a ball this product must be multiplied by the sine of the angle between the direction of motion of the ball relative to the air and the axis of spin. Rayleigh's law was adopted by P. G. Tait (*Nature*, June 29, 1893; Papers, vol. 2, p. 386) in his mathematical calculation of the trajectories of a golf ball and by Sir J. J. Thomson in his experimental method (*Nature*, Dec. 22, 1910) of imitating these trajectories by means of the path of an electron in superposed electric and magnetic fields. A slightly different law is adopted by Appell (*Journ. de Physique*, vol. 7 (1917), p. 5) in his analysis of Carrière's experiments, ibid., vol. 5 (1916), p. 175. The formula has been established for a cylindrical surface of arbitrary shape in a stream of fluid on the assumption that there is a circulation round the cylinder. This is the basis of the theory of sustentation developed by Kutta, Joukowsky and others (see Joukowsky, Aérodynamique, Paris, 1916, and a paper by R. Jones, *Proc. Roy. Soc. London*, vol. 92, A (1916), p. 107) and of the theory of propeller action which has been developed by R. Grammel (*Jahrb. d. Schiffsbautechnik*, vol. 17 (1916), p. 367).

† Royal Institution, March 18 (1910); *Nature*, Dec. 22 (1910).

effect of rotation has been used to increase the range of spherical projectiles and in his experiments with golf balls* Tait conclusively proved that the great factor in long driving was the underspin communicated to the ball by the impact of the club.

Lafay† has recently determined the way in which the pressure varies over the surface of a cylinder rotating rapidly in a stream of air and finds that when the transverse force is in the direction indicated by Magnus it is produced chiefly by the suction on the side which is moving in the same direction as the stream of air.‡ The analogy with the wing of an aeroplane is thus complete.

The distribution of pressure is important, for it should indicate whether or not a rotating cylinder carries along with it one or more vortices that do not produce any circulation around the cylinder.

Lafay's results for a speed of rotation of 9,450 revolutions per minute and a velocity of the air stream of 19 meters per second are given below (p) and compared with the corresponding results for the case of no rotation (p_0). The pressures are in millimeters of water and represent deviations from the atmospheric pressure.

θ	0	15	30	45	60	75	90
p	−18	−16	−12	− 9	− 8	− 5	− 2
p_0	−10	−10	−10	−10	−11	− 13	− 17

	105	120	135	150	165	180
	3	8	14	20	22	18
	− 21	− 21	− 13	7	20	23

θ	− 0	−15	−30	−45	−60	− 75	− 90
p	−18	−23	−32	−63	−93	−102	−104
p_0	−10	−10	−10	−11	−12	− 14	− 17

	−105	−120	−135	−150	−165	−180
	− 85	− 63	− 37	− 14	7	18
	− 20	− 21	− 13	7	19	23

* *Badminton Magazine*, March (1896).

† *Comptes Rendus*, vol. 153 (1911), p. 1472.

‡ The experiments and remarks made by W. S. Franklin, *Journ. Franklin Inst.*, vol. 177 (1914), p. 23, are of some interest in this connection.

To compare these results with formula (3), we must take into account the fact that the theory gives no drag X when $k = 0$ and $c = 0$. It seems reasonable to suppose, however, by analogy with (3) that the correct formula is of type

$$u^2 + v^2 = \left[f(\theta) + \frac{k}{a} + \frac{c}{a} \frac{a^2 - r_0^2}{a^2 - 2ar_0 \cos(\theta - \theta_0) + r_0^2} \right]^2,$$

where $f(\theta)$ is some function which gives the distribution of velocity in the case when there is no rotation of the cylinder. Now if c were zero the above formula would indicate that $\sqrt{P - p}$ should differ from $\sqrt{P - p_0}$ by a constant proportional to k/a, where P is some constant. This means that

$$(p - p_0)^2 + \lambda(p + p_0)$$

should be constant where λ is some constant. This, however, is far from the case as may be seen from some of the values of $(p - p_0)^2$ and $p + p_0$

$(p - p_0)^2$	64	36	4	1	9	8100	441	25
$p + p_0$	−28	−26	−22	−21	−29	−118	−7	41

It seems reasonable then to assume that the flow is modified by the presence of one or more vortices and if we wish to try to account for the drag on the cylinder with the aid of these vortices it is necessary to assume that they are in motion relative to the cylinder just as in Kármán's theory of resistance.

It should be noticed that the region of low pressure in Lafay's experiment occurs in the neighborhood of $\theta = -90°$, and so is directly opposite to the region where a vortex can remain in stable equilibrium. Hence if a vortex forms in the region of low pressure* it cannot remain stationary.

If c is negative and u_0 positive so that the vortex is carried away by the stream, Y may be decreased in magnitude and may even be positive instead of negative. A vortex with counterclockwise rotation may perhaps form when U is greater than the circumferential velocity of the rotating cylinder. As before, we suppose that the rotation of the

* Lord Kelvin pointed out that if the velocity of a spherical solid moving through a fluid exceeds a certain value the pressure becomes negative when calculated by the ordinary theory and so cavitation must commence at the back of the sphere; coreless vortices will be formed periodically and shed off behind the sphere during its motion through the fluid. *Phil. Mag.*, vol. 23 (1887).

cylinder is counterclockwise. If on the other hand c is positive and u_0 positive our formula shows that Y is negative and numerically greater than in the case when there is no vortex. A vortex with clockwise rotation may, perhaps, form in the neighborhood of $\theta = -90°$ when the circumferential velocity of the rotating cylinder is greater than the velocity of the stream.

To account for the force on the cylinder in the direction of the axis of x it is necessary to suppose that a vortex with counterclockwise rotation moves away from this axis and that a vortex with clockwise rotation moves towards this axis.

It should be mentioned that Lafay has found* that the direction of the transverse force in the Magnus experiment could be reversed. Experimenting with a smooth aluminum cylinder 35 cm. long and 10 cm. in diameter, he found that if the velocity of the air stream were kept constant at 18 or 19 meters per second and the velocity of rotation gradually increased, the direction of the force on the cylinder first swung to one side of the air stream, attained a maximum inclination to it of about 11°, then swung to the other side, attained a maximum inclination of about 57°, and finally appeared to approach asymptotically to a direction making an angle of 45° with the air stream.

The maximum inverse effect occurred when the cylinder was rotating at a speed of 1570 turns per minute giving a circumferential velocity of about 8.22 meters per second which is less than the velocity of the air stream. On the other hand the direct effect was quite marked when the speed of rotation was 9450 turns per minute, in which case the circumferential velocity is in the neighborhood of 50 meters per second and is greater than the velocity of the air stream. This is exactly in accordance with the above view that the transverse force is modified by the production of vortices in the neighborhood of the region $\theta = -90°$ and that the direction of rotation in the vortices depends upon whether the circumferential velocity is greater or less than that of the stream of air.

It is well known that the drift of a projectile fired from a rifled gun is exactly opposite to the direction of the transverse force which is indicated by the normal Magnus effect, but here we are dealing with a case in which the component velocity of translation of the bullet in a direction perpendicular to its axis is perhaps at some time greater than the circumferential

* *Comptes Rendus*, vol. 151 (1910), p. 867; vol. 153 (1911), p. 1472.

velocity due to its spin and so the transverse force may be reversed as in Lafay's experiments. In Kármán's theory of resistance* it is supposed that vortices with opposite senses of rotation are formed alternately behind a cylinder and move down the stream in two rows at a certain distance apart. These vortices occupy the region of the "wake" behind a cylinder in a stream of fluid. Now in Lanchester's theory of the Magnus effect† it is assumed that the wake behind the cylinder is displaced to one side on account of the rotation of the cylinder. If the vortices in this wake are produced somewhere in the neighborhood of $\theta = -90°$, the displacement of the wake to one side would be accounted for by the previous remark that a vortex with counterclockwise rotation moves away from the axis of x and that a vortex with clockwise rotation moves towards‡ the axis of x while it is carried down the stream.§ It should be mentioned, however, that here we imagine the resistance of the cylinder to arise from the velocities of the vortices in a direction at right angles to the stream, while in Kármán's theory the resistance arises from the momentum which is carried away from the cylinder whenever a fresh pair of vortices is formed. This momentum is calculated from the circulations around the vortices and the distance between the two rows, while the rate at which the vortices are formed is calculated from their final distance apart in a row and their final velocity relative to the cylinder which is now parallel to the axis of x. The difference between the two points of view is probably the same as the difference between the initial and final stages of an action; for, when the motion of the vortex perpendicular to the stream is considered, we are dealing with the actual transfer of momentum from the cylinder to the fluid or vice-versa.

§ 2. *Two Rectilinear Vortices and a Rotating Circular Cylinder in a Stream of Fluid.*

An interesting attempt to throw light on the initial stages of the formation of the two rows of vortices in the Kelvin-

* *Phys. Zeitschr.* (1912), *Gött. Nachr.* (1911). See also Joukowsky Aérodynamique, Paris (1916), Ch. 8; Lamb, Hydrodynamics, 4th ed. (1916), p. 219.

† Aerodynamics, vol. I.

‡ When vortices originate on opposite sides of the x-axis as in the case of no rotation they must both move away from the axis of x if they are to produce a positive x.

§ If it crosses the axis of x it must then move away from it.

Kármán theory of resistance has been made by L. Föppl* who has shown that when a cylinder is at rest in a stream of fluid two vortices which are images of one another in the axis of x can be in equilibrium in a stationary position behind the cylinder provided they lie on the curves

$$\pm 2r^2 \sin^2 \theta = r^2 - a^2.$$

Föppl found that the vortices are stable for symmetrical displacements but unstable for asymmetric displacements. The former result is of some meteorological interest in connection with the flow of air past a mountain or other obstacle which is shaped roughly like a half cylinder standing on a plane. It is well known in fact that an eddy can form behind a mountain over which a wind is blowing†; near the rock of Gibraltar the eddy motion is sometimes quite large.

The second result is of interest because it indicates that if two vortices form in symmetrical positions behind a cylinder as they do behind a flat plate, they will be in unstable equilibrium for asymmetric displacements; consequently one vortex may be imagined to get ahead of the other and a new one to be formed to take its place, thus giving rise to an alternate formation as imagined by Lord Kelvin and Kármán.‡ It should be mentioned that in Föppl's analysis the strength of the vortices when in equilibrium increases with their distance from the cylinder while in Kármán's analysis the strengths of all the vortices in one row are supposed to be the same and equal but opposite in sign to those of the vortices in the other parallel row.

If $z_0 = r_0 e^{i\theta_0}$ indicates the position of one of the vortices and c the strength, we have in Föppl's case

$$c = U\left(1 - \frac{a^2}{r_0^2}\right)^2 \left(r_0 + \frac{a^2}{r_0}\right)$$

* *München Sitzungsberichte* (1913).

† See, for instance, W. H. Dines, Report of the Advisory Committee for Aeronautics, No. 92, March (1913), W. N. Shaw, *Science Progress*, vol. 6, p. 345.

‡ This alternate formation of two rows of vortices has been observed on many occasions. See for instance Osborne Reynolds, *Phil. Trans.*, vol. 174 (1883); Ahlborn, "Ueber den Mechanismus des hydrodynamischen Widerstandes," Hamburg (1902); Mallock, *Proc. Roy. Soc.*, vol. 79 (1907), p. 262; vol. 84 (1910), p. 490; *Engineering*, April 19 (1912); H. Bénard, *Comples Rendus*, vol. 147 (1908), pp. 839, 970; vol. 156 (1913), p. 1003; vol. 157, pp. 7, 89, 171; Rayleigh, *Phil. Mag.*, vol. 29 (1915), p. 433. Joukowsky, loc. cit.

and the velocity at the surface of the cylinder is given by the formula

$$u^2 + v^2 = \left[2U \sin\theta - \frac{4cr_0(r_0^2 - a^2)\sin\theta \sin\theta_0}{\{r_0^2+a^2-2ar_0\cos(\theta-\theta_0)\}\{r_0^2+a^2-2ar_0\cos(\theta+\theta_0)\}} \right]^2.$$

Since $2r_0^2 \sin\theta_0 = r_0^2 + a^2$ it appears that the velocity vanishes when $\theta = 0$, $\theta = \pi$ and also when

$$\cos\theta = \frac{1}{2a}\left(r_0 + \frac{a^2}{r_0}\right)\cos\theta_0 - \frac{1}{a}\sqrt{3r_0^2 + 4a^2}\sin^2\theta_0.$$

This equation can give a real value of θ, for when $r_0^2 = 3a^2$, we have

$$\cos\theta = \frac{4\sqrt{6} - \sqrt{13}}{9} < 1.$$

When the surface velocity q is plotted as a function of θ, q and θ being regarded as polar coordinates, a curve is obtained which is shaped like a butterfly with two wings. The measurements of J. T. Morris,* A. Thurston† and A. Lafay‡ indicate that for a cylinder in a stream of fluid the curve indicating the distribution of velocity should be shaped like a butterfly with only one pair of wings. The lack of agreement is to be expected on account of the instability of the two vortices behind the cylinder. A comparison of the theoretical formula ought to be made with some measurements of the velocity over the surface of a semi-cylinder standing on a smooth plane surface over which a wind is blowing. Of course the roughness of a rotating ball or cylinder has a great influence on the magnitude of the transverse force exerted by the wind as is clearly shown in the experiments of Sir Ralph Payne-Gallwey,§ Sir Joseph Thomson and Commandant Lafay. Curiously enough the former found that to obtain the best lifting effect with a golf ball the ball must not be too rough. This suggests that with a very rough ball vortices are produced which modify the Magnus effect.

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* *Engineering*, Aug. 8, 1913.
† Ibid., Aug. 21, 1914.
‡ Loc. cit.
§ *The Times*, March 16, 23, 1909. *Nature*, April 22, 1909.

It may be of interest to indicate briefly the extension of Föppl's analysis for the case in which the cylinder rotates and there is no symmetry about the axis of x. Using z_0 and z_2 to indicate the positions of the vortices and z_1 and z_3 those of their images in the cylinder, the appropriate expressions for ϕ and ψ are given by

$$\phi+i\psi=U\left(z+\frac{a^2}{z}\right)+ik\log z+ic\log\frac{z-z_0}{z-z_1}-ic\log\frac{z-z_2}{z-z_3}.$$

Differentiating to obtain the velocities we have

$$u-iv=U\left(1-\frac{a^2}{z^2}\right)+\frac{ik}{z}+ic\left[\frac{1}{z-z_0}-\frac{1}{z-z_1}\right]-ic\left[\frac{1}{z-z_2}-\frac{1}{z-z_3}\right],$$

$$u_0-iv_0=U\left(1-\frac{a^2}{z_0^2}\right)+\frac{ik}{z_0}-ic\left[\frac{1}{z_0-z_1}+\frac{1}{z_0-z_2}-\frac{1}{z_0-z_3}\right],$$

$$u_2-iv_2=U\left(1-\frac{a^2}{z_2^2}\right)+\frac{ik}{z_2}+ic\left[\frac{1}{z_2-z_0}-\frac{1}{z_2-z_1}+\frac{1}{z_2-z_3}\right].$$

Writing

$$z_0=r_0e^{i\theta_0},\quad z_2=r_2e^{i\theta_2},\quad R^2=r_0^2+r_2^2-2r_0r_2\cos(\theta_0-\theta_2),$$

$$S^2=r_0^2r_2^2+a^4-2a^2r_0r_2\cos(\theta_0-\theta_2)$$

we find that when $u_0=v_0=u_2=v_2=0$,

$$0=U\left(1-\frac{a^2}{r_0^2}\cos 2\theta_0\right)+\left(\frac{k}{r_0}-\frac{cr_0}{r_0^2-a^2}\right)\sin\theta_0+\frac{c}{R^2}(r_2\sin\theta_2-r_0\sin\theta_0)+\frac{cr_2}{S^2}(r_0r_2\sin\theta_0-a^2\sin\theta_2),$$

$$0=U\frac{a^2}{r_0^2}\sin 2\theta_0+\left(\frac{k}{r_0}-\frac{cr_0}{r_0^2-a^2}\right)\cos\theta_0$$

$$+\frac{c}{R^2}(r_2\cos\theta_2 - r_0\cos\theta_0) + \frac{cr_2}{S^2}(r_0r_2\cos\theta_0 - a^2\cos\theta_2)$$

and two similar equations with the suffixes 0 and 2 interchanged and $-c$ written in place of c.

Multiplying the first of these equations by $\cos\theta_0$, the second by $\sin\theta_0$ and subtracting, we get

$$(r_0^2 - a^2)\left[\frac{U}{r_0^2}\cos\theta_0 - \frac{cr_2}{R^2S^2}(r_2^2 - a^2)\sin(\theta_0 - \theta_2)\right] = 0.$$

Similarly

$$(r_2^2 - a^2)\left[\frac{U}{r_2^2}\cos\theta_2 - \frac{cr_0}{R^2S^2}(r_0^2 - a^2)\sin(\theta_0 - \theta_2)\right] = 0,$$

$$\left[r_0 - \frac{a^2}{r_0}\right]\cos\theta_0 = \left(r_2 - \frac{a^2}{r_2}\right)\cos\theta_2.$$

This equation tells us that the projection on the axis of x of the interval between a vortex and its image in the cylinder is the same in both cases. It is clear then that the vortex which is furthest from the plane $y = 0$ is also furthest from the axis of the cylinder. There is another equation connecting r_0, θ_0, r_2, θ_2, but it is very complicated; the two equations show, however, that when one vortex is given the position of the other is determined. The component forces on the cylinder are easily found by an extension of the analysis of § 1. They are

$$X = 2\pi\rho c(v_0 - v_1 - v_2 + v_3),$$

$$Y = 2\pi\rho(Uk - cu_0 + cu_1 + cu_2 - cu_3).$$

When both vortices are stationary we have $X = 0$, $Y = 2\pi\rho Uk$ as before. This result seems to be true for any number of stationary vortices outside a cylinder, as is probably well known.*

We have seen that a rotation of the cylinder alters the possible stationary positions for a pair of vortices; it may also alter the period of formation of the vortices in Kármán's theory of resistance. This is not easy to settle mathematically but the matter may perhaps be tested experimentally.

Eiffel found during his measurements of the force exerted by a stream of air on a sphere that at a speed above twenty

* See for instance the remark in § 3 of Föppl's paper.

miles an hour the flow was smoother and the force more constant. The vortices behind the sphere are usually said to be flattened out* and presumably the period of formation which depends on the distance between the two rows of vortices is changed. The rate of formation probably increases with the velocity of the stream until the flow becomes practically steady at high speeds.†

Now a rotation of the cylinder or sphere may cause a change in the critical velocity above which the flow is practically steady, consequently it may be worth while to determine this critical velocity for a given velocity of the stream and different velocities of rotation of the cylinder. This critical velocity may be closely connected with Lafay's critical velocity at which the sign of the transverse force in the Magnus effect is reversed; it should be noticed, however, that Lafay made his experiments in a wind of 19 meters a second which had a velocity more than double Eiffel's critical velocity and noticed at which *speed of rotation* the change occurred.

It is known that at high speeds the periodic formation of vortices is responsible for the production of sound‡ and the period is presumably that of the sound. In the case of the aeolian harp§ the sound is most intense when the period is close to one of the natural periods of the stretched string or wire. An effect of rotation of the stretched string or wire on the pitch of the sound produced in a given type of wind might perhaps be determined experimentally but there would be difficulties. For the mathematical theory of the aeolian harp the sound produced by the oscillation of a vortex about a state of uniform motion ought also to be considered, for this may contribute to the observed sound as well as periodic formation of vortices, but the effect is probably negligible.

§ 3. *Vortices in a Compressible Fluid.*

It was shown by Lord Kelvin that vortex lines in a compressible fluid move with the fluid provided the density of

* Cf. Loening, Military Aeroplanes, p. 51.

† Cf. Cowley and Levy, Aeronautics in Theory-Experiment (1918), pp. 17–20.

‡ See for instance Mallock, *Proc. Roy. Soc. London*, vol. 84 (1910), p. 490.

§ Lord Rayleigh, Theory of Sound, vol. II, p. 412, vol. 1, p. 212; *Phil. Mag.*, vol. 29 (1915), p. 433.

the fluid is a function of the pressure only and the body forces have a single-valued potential. It is also true that the circulation in any circuit moving with the fluid remains constant. An accurate theory of vortex motion in a compressible fluid is difficult. To make progress, it seems worth while to make an assumption which is nearly true except in the immediate neighborhood of a vortex.

Let us consider a two-dimensional irrotational motion in which the velocity potential ϕ satisfies the wave equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2},$$

wherein c, the velocity of sound in the fluid, is supposed to be constant. The component velocities (u, v) will then satisfy the same equation. A solution appropriate for the representation of a rectilinear vortex moving with constant component velocities a, b may be derived from the well known solution for a stationary vortex by an application of the transformations of the theory of relativity. The result is

$$u=-\frac{\mu c(y-bt)\sqrt{c^2-a^2-b^2}}{(c^2-b^2)(x-at)^2+(c^2-a^2)(y-bt)^2+2ab(x-at)(y-bt)},$$

$$v=\frac{\mu c(x-at)\sqrt{c^2-a^2-b^2}}{(c^2-b^2)(x-at)^2+(c^2-a^2)(y-bt)^2+2ab(x-at)(y-bt)},$$

where $2\pi\mu$ is the strength of the vortex.*

To find the paths of the particles of fluid relative to the moving vortex, we write $X=x-at$, $Y=y-bt$, $X^2+Y^2=z$, $bX-aY=w$, $u=dx/dt$, $v=dy/dt$; then it is easy to see that

$$\frac{1}{2}\frac{dz}{dt}+aX+bY=0, \qquad \frac{dw}{dt}+\mu\frac{c\sqrt{c^2-a^2-b^2}(aX+bY)}{c^2z-w^2}=0,$$

$$\therefore \quad \frac{dz}{dw}=\frac{2(c^2z-w^2)}{\mu c\sqrt{c^2-a^2-b^2}},$$

$$\therefore \quad z=Ae^{2cw/\mu\sqrt{c^2-a^2-b^2}}+\frac{1}{c^2}\left[w^2+\mu\frac{w}{c}\sqrt{c^2-a^2-b^2}+\frac{\mu^2}{2c^2}(a^2-c^2-b^2)\right],$$

where A is an arbitrary constant.

* μ is positive for counterclockwise rotation.

Thus relative to the moving vortex a particle of fluid appears to describe a curve whose equation is

$$X^2 + Y^2 = Ae^{\frac{2\pi}{\mu}(bX - aY)/\mu\sqrt{c^2-a^2-b^2}} + \frac{1}{c^2}\Big[(bX - aY)^2 + \frac{\mu}{c}\sqrt{c^2 - a^2 - b^2}\,(bX - aY) + \frac{\mu^2}{2c^2}(c^2 - a^2 - b^2)\Big].$$

When $A = -(\mu^2/c^2)(c^2 - a^2 - b^2)$, we obtain an equation which is satisfied by $X = 0$, $Y = 0$ and this represents a curve having an isolated point at the origin which is in accordance with the theorem that the vortex moves with the fluid. As A varies, we obtain first a number of ovals surrounding the origin, then a curve which touches itself on the axis of y,* after completing the circuit, and then goes to infinity. Finally we obtain a series of curves which lie outside the last one and like it go to infinity. Each curve is described by a point which starts moving almost in the direction of the x-axis, then swings around the origin in the clockwise direction and finally returns to a state of motion very nearly parallel to the x-axis.

Let us now consider two rectilinear vortices of strengths $2\pi\mu$ and $-2\pi\mu$ respectively moving parallel to the axis of x with constant velocity a and at a distance apart equal to $2y$. If the velocity of each vortex is that produced by the other, we have simply

$$a = \frac{\mu c}{2y\sqrt{c^2 - a^2}}$$

or

$$a^2 = \tfrac{1}{2}c^2 \pm \sqrt{\tfrac{1}{4}c^2 - \frac{\mu^2 c^2}{4y^2}}.$$

For a real value of a^2 we must have $c^2y^2 > \mu^2$. Hence when the circulations around the vortices are given they cannot move parallel to one another at a distance apart less than $2\mu/c$. If the two vortices are in a stream of fluid moving with velocity U the equation determining a is

$$a = U + \frac{\mu c}{2y\sqrt{c^2 - a^2}}$$

* We put $y = bX - aY$, $x = aX + bY$ to obtain a curve which is symmetrical with regard to the axis of y.

or

$$4y^2(c^2 - a^2)(a - U)^2 = \mu^2 c^2.$$

It should be noticed that a is numerically less than c whatever the value of U.

The velocity potential of a vortex moving with constant velocity differs by a constant from the function

$$\mu \tan^{-1} \frac{(ay - bx)\gamma}{a(x - at) + b(y - bt)} \equiv \mu \int_{-\infty}^{t} \frac{ds}{\sqrt{t - s}} B(x, y, s),$$

where

$$\gamma = \left(1 - \frac{a^2 + b^2}{c^2}\right)^{1/2}$$

and

$$A(x, y, s) + iB(x, y, s) = \left[\frac{a^2 + b^2}{a(x - as) + b(y - bs) - i\gamma(ay - bx)}\right]^{1/2},$$

$$A(x, y, s) + iB(x, y, s) = \left[\frac{a^2 + b^2}{a(x - as) + b(y - bs) - i\gamma(ay - bx)}\right]^{1/2}$$

For a vortex which is moving with component velocities $\xi(\tau)$, $\eta(\tau)$ at time τ and is at an infinite distance from the origin at time $\tau = -\infty$, the natural generalization of the function $B(x, y, s)$ is obtained as follows:

Let τ be defined in terms of x, y, s by the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 = c^2(s - \tau)^2 \quad \tau \leqq s,$$

where $\xi'^2(\tau) + \eta'^2(\tau) < c^2$ and primes denote differentiations with respect to τ. Let

$$l(\tau) = \xi'(\tau) + i\sigma\eta'(\tau), \qquad c^2 p(\tau) = \xi'^2 + \eta'^2,$$

$$m(\tau) = \eta'(\tau) - i\sigma\xi'(\tau), \qquad c^2\sigma^2 = c^2 - \xi'^2 - \eta'^2;$$

then

$$A(x, y, s) + iB(x, y, s) = \left[\frac{c^2 p(\tau)}{\{x - \xi(\tau)\}l(\tau) + \{y - \eta(\tau)\}m(\tau) - c^2(s - \tau)p(\tau)}\right]^{1/2}.$$

This result may be of interest for an analysis of the sound produced when a vortex oscillates about a state of uniform motion. It must be remembered, however, that the above analysis is only approximate, for velocities are treated as small in the derivation of the wave equation.

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SHORTER NOTICES.

Lectures on the Philosophy of Mathematics. By JAMES BYRNIE SHAW. Chicago, The Open Court Publishing Company, 1918. viii + 206 pp.

THE purpose of Professor Shaw's book is a discussion of the evergreen question: What is mathematics? While in his first chapter the author develops in a highly exalted style various aspects of this subject, the greater part of the subsequent chapters may be said to be essentially devoted to two more specific questions, viz., what influences operate and have operated in the development of mathematics, and how may existing mathematics be concisely described. With the treatment of these questions, perhaps not always recognized as explicit and distinct, Chapters II to XIII are taken up, together occupying 140 pages. To the first question the author gives a positive answer, viz.: "Mathematics is a creation of the mind and is not due to the generalization of experiences or to their analysis; nor is it due to an innate form or mold which the mind compels experience to assume, but is the outcome of an evolution, the determining factors of which are the creative ability of the mind and the environment in which it finds the problems which it has to solve in some manner and to some degree." The second question is answered in a negative sense; as the various fields and principles of mathematics are discussed, the conclusions are reached that mathematics is not wholly arithmetic, nor geometry, nor logistic; that mathematics can not be completely characterized as a theory of invariance, nor as a theory of functions, etc., however important each of these principles may be. The closest approach to a satisfactory answer to our second

question the author appears to find in the statement that mathematics is a theory of equations. On page 152 we read: "This is the most important central principle of mathematics, namely, that of inversion, or of creating a class of objects that will satisfy certain defining statements. If the mathematician does not find these at hand in natural phenomena, he creates them and goes on in his uninterrupted progress. This might be considered to be *the* central principle of mathematics, for with the new creation we start a new line of mathematics, just as the imaginary started the division of hypernumbers, just as the creation of the algebraic fields started a new growth in the theory of numbers."

Chapters XIV, XV and XVI take up respectively "Sources of mathematical reality" (15 pages), "The methods of mathematics" (17 pages) and "Validity of mathematics" (10 pages). There is finally given on pages 196 and 197 a double entry table for the classification of existing mathematics, presenting in schematic form the division of the whole field on the basis of which the discussion is carried on in Chapters II to XIII. This very interesting and suggestive mode of dividing mathematics is not the least valuable part of the book. Each mathematical topic is classified as to its structure and as to its central principle. Structurally, mathematics falls into two main divisions, static mathematics and dynamic mathematics. In the former group we find arithmetic, geometry, tactic and logistic; in the latter group are placed algorithms, algebra, transmutations and inferences. The central principles of mathematics are taken to be form, invariance, functionality and inversion.

The book is written in an easy, flowing style; sometimes, as for instance in the first six pages in a flowering style, reminiscent of an earlier literary epoch. In places it is sketchy, so much so as to be of little value to the non-professional reader for whom the book is intended, and to be somewhat irritating to the mathematician. A rather strong example of this is found on page 26, where we find the following statement: "By introducing a 'measure of a set' Lebesgue and others have found a means of handling sets satisfactorily"; others are to be found on pages 90, 91, and at the end of chapter VII. This floweriness and sketchiness of style have in a few instances led to what seems to the reviewer to be an overstatement of perfectly sound positions. As such we

would characterize the assertion that "Nothing whatever in our sense-data tells us that the earth is rotating" (page 156), and the claim that "we find in mathematics that subject whose results have lasted through the vicissitudes of time and are regarded universally as the most satisfactory truths the human race knows" (page 154). Why is Lie relegated to the remote period of the Norsemen?

While the author "cherishes the hope that the professional philosopher too may find some interest in these lectures," he addresses himself primarily to "students of fair mathematical knowledge," such as may be secured through an ordinary college course in mathematics. The book is intended to be non-technical from the philosophical as well as from the mathematical point of view. Indeed it seems to the reviewer that the author has taken the term "philosophy of mathematics" in the more or less popular sense, in which it means general discussion about the nature of mathematics, rather than in the sense of systematic, rigorous, scientific discussion. This being assumed, and the assumption is not intended as a criticism of the value of the book, it is understood why one finds nowhere a statement of the author's general philosophical position, of his beliefs and convictions on some of the fundamental questions of philosophy, on which many of his conclusions are based, and a knowledge of which is indispensable for an understanding and just appraisal of his work. Hints are found scattered throughout the book, as for instance on pages 30, 55, 59, 88, etc.; and perhaps the most definite statement of a philosophical credo is that on page 166: "The mind, it is true, as Kant insisted, organizes experience, but it does this by methods that are evolutionary. It originates schemes from its own activity, and makes a choice of which of several equally valid schemes it will use." But it seems clear that a systematic, critical examination of the philosophical position and of the conclusions derived therefrom would be entirely out of keeping with the purpose of this book.

While granting without hesitation, particularly at the present stage of development of the subject, the value of a general, let us say intuitional, discussion of the philosophy of mathematics, the reviewer would like to use this opportunity to call attention to the scientific aspects of this subject. To these may be said to belong (*a*) the study of the logical structure of mathematics; (*b*) the philosophical discussion of concepts to

which mathematics gives rise, such as infinity, countability; (c) the introduction of a mathematical point of view, and as far as possible, of mathematical methods into the discussion of problems belonging to the domains of epistemology, of ethics, and to other philosophical fields. For progress in the treatment of these questions the closest cooperation between philosophers and mathematicians is essential. Such cooperation will certainly be stimulated by the reading of Professor Shaw's book. ARNOLD DRESDEN.

Functions of a Complex Variable. By THOMAS M. MACROBERT. London and New York, Macmillan, 1917. xiv + 298 pp.

THIS book is designed for students who have acquired a good working knowledge of the calculus and desire to become acquainted with the theory of functions of a complex variable and its principal applications. The material has been well chosen for accomplishing this purpose. The first two chapters are intended to familiarize the student with the geometrical representation of complex numbers in a plane and with simple rational and irrational functions of a complex variable. Some fundamental properties of holomorphic functions are established in Chapter III and these are used in defining certain elementary transcendental functions. Chapters IV to VII deal in order with the theory and practice of integration, convergence of series and the Taylor and Laurent expansions, uniformly convergent series and infinite products, and various summations and expansions. Up to this point (page 131) the book gives to the student an introduction to the elementary parts of the classic theory of functions of a complex variable with some of the simpler applications.

The theory of the gamma functions is presented in Chapter VIII (pages 132–159) in a way which is not very satisfying. The development of the initial elementary properties of elliptic functions is to be found in Chapters IX, X, XI (pages 160–207) in an exposition which is pleasing and particularly well suited to the needs of the young learner. The remaining chapters (IX to XV, pages 208–276) are devoted to differential equations, in part to general theorems and in other part to special cases, such as the equations of Legendre and Bessel. This division of the book also is well suited to the needs of the beginner.

Besides numerous worked examples the book contains many

problems. In addition to those interspersed throughout the text, there are to be found at the ends of the chapters and in a set of miscellaneous examples at the end of the book no fewer than 550 problems. Many of these are quite elementary in character and serve merely to familiarize the reader with the notions involved in the text. Many others contain significant theorems which would be demonstrated in a more extensive treatise.

Concerning the character of his exposition the author says in his preface: "In order to avoid making the subject too difficult for beginners, I have abstained from the use of strictly arithmetical methods, and have, while endeavoring to make the proofs sufficiently rigorous, based them mainly on geometrical conceptions." It is evident that the desirable golden mean here is a matter on which there is likely to be difference of opinion. The reviewer believes that the statements are sometimes too loose. Thus "closed region" is first defined on page 92. Yet several times on the earlier pages statements have been made involving the word "region" which are correct only when one understands "closed region." For cases in point see page 24, line 6 up; page 26, line 7; page 39, line 7 up. In theorem 2 of page 93 it should be specified that the path of integration is finite in length or the proof should be modified. For theorem 2 on page 39 one should read "No [closed] region can contain an infinite number of isolated singularities [and no other singularities]," the words in brackets being those which one must insert into the theorem to make it accurate. These may be taken as examples of loose statements which are all too frequent, especially in the first 131 pages, in which the more general matters are treated.

A student who has been forewarned against these somewhat loose statements will find the book one by the reading of which he will be much profited. Many persons will probably find it also a suitable source of elegant elementary problems for enforcing a clear understanding of numerous fundamental notions and theorems. R. D. CARMICHAEL.

Leçons sur les Fonctions Elliptiques en Vue de leurs Applications. Par R. DE MONTESSUS DE BALLORE. Paris, Gauthier-Villars, 1917.

THE object of these lectures, which M. de Montessus de Ballore delivered at the Faculté des Sciences of Paris in 1915–

1916, is to present in an elementary manner the fundamental properties of the ordinary elliptic functions. The first part deals with the Jacobian functions sn, cn, dn, and the elliptic integrals, the second part with the Weierstrassian forms $\wp u$, ζu, σu, and the third part with the general properties of elliptic functions and their applications to the Jacobian and Weierstrassian forms.

In the fourth and last part, the θ-functions are considered.

As may be expected from the announcement in the preface, the book does not contain anything beyond well known elementary propositions. In view of possible applications, of which the book does not contain any, particular attention is given to the construction of formulas for numerical computations.

The reviewer must confess that he would have expected a course of lectures on elliptic functions at the University Paris planned from a higher point of view, as is customary at that famous center of mathematical activity. For this reason the Leçons, in spite of their excellence of execution and typography, are disappointing.

ARNOLD EMCH.

NOTES.

THE April number (volume 41, number 2) of the *American Journal of Mathematics* contains the following papers: "Asymptotic satellites near the straight-line equilibrium points in the problem of three bodies," by DANIEL BUCHANAN; "Concerning the invariant theory of involutions of conics," by WAYNE SENSENIG; "Note on seminvariants of systems of partial differential equations," by A. L. NELSON; "On a method for determining the non-stationary state of heat in an ellipsoid," by BIBHUTIBHUSAN DATTA; "Nilpotent algebras generated by two units, i and j, such that i^2 is not an independent unit," by G. W. SMITH.

THE Circolo matematico di Palermo announces that it has resumed the publication of its *Rendiconti*. Contributors are invited to submit manuscripts to its committee of publication under the usual conditions.

BEGINNING with the volume for 1919, *Nyt Tidsskrift for Matematik* will be published by the Mathematical society of Copenhagen. The name of the periodical has been changed to *Matematisk Tidsskrift*.

AT the meeting of the London mathematical society held March 13, the following papers were read: By J. HAMMOND, "The solution of the quintic"; by L. J. MORDELL, "A simple algebraic summation of Gauss's sums"; by P. A. MACMAHON, "Divisors of numbers and their continuations in the theory of partitions"; by S. RAMANIJAN, "Convergence properties of partitions," and "Algebraic relations between certain infinite products."

AT the meeting of the Edinburgh mathematical society held March 14, the following papers were read: By C. G. KNOTT, "Systems of rays in quaternion symbolism"; by E. T. WHITTAKER, "Some disputed questions of probability."

THE British association for the advancement of science will resume the holding of summer meetings. In 1919 the meeting will be held at Bournemouth under the presidency of Sir CHARLES PARSONS.

THE Adams prize for 1918 has been awarded to Professor J. L. NICHOLSON, of King's College, London, for his researches in mathematical astronomy.

DR. G. WALLENBERG, of the Charlottenberg technical school, has been promoted to an honorary professorship of mathematics.

PROFESSOR O. HÖLDER has been chosen rector of the University of Leipzig.

ASSOCIATE professor W. MATTHIES, of the University of Basle, has been promoted to a full professorship of mathematical physics.

PROFESSOR G. ROST has been chosen rector of the University of Würzburg.

PROFESSOR J. K. LAMOND, of Pennsylvania College, Gettysburg, has been given two years leave of absence to engage in Red Cross work. He is now associate director of the department of military relief, in charge of home service, Pennsylvania-Delaware division.

DR. C. E. WILDER has been appointed instructor in mathematics at Clark University.

PROFESSOR W. H. GARNETT, of Wesleyan College, Winchester, Ky., is on leave of absence for the year.

PROFESSOR A. L. RHOTON, of the department of mathematics at Georgetown, Ky., College, has been appointed associate professor of education at Pennsylvania State College.

DR. G. W. MULLINS, of Barnard College, Columbia University, has been promoted to an assistant professorship of mathematics.

MR. W. W. ELLIOTT has been appointed instructor in mathematics at the University of Kentucky.

PROFESSOR H. G. ZEUTHEN, of the University of Copenhagen, died February 15, 1919, at the age of eighty years.

PROFESSOR F. E. MILLER, head of the department of mathematics at Otterbein College, died March 26, 1919. He had been a member of the American Mathematical Society since 1910.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BROUWER (L. E. J.). Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. 1ter Teil: Allgemeine Mengenlehre. Amsterdam, 1918. M. 2.00

BURNSIDE (W. S.) and PANTON (A. W.). The theory of equations. London, Longmans. Volume 1, 8th edition, 1918. 14 + 286 pp. Volume 2, 6th edition, 1916. 9 + 288 pp. Each, 10s. 6d.

FUETER (R.). Synthetische Zahlentheorie. Leipzig, Göschen, 1917. 8 + 271 pp.

GOURSAT (E.). Cours d'analyse mathématique. Tome II: Théorie des fonctions analytiques. Équations différentielles. 3e édition. Paris, Gauthier-Villars, 1918. 8vo. 4 + 670 pp. Fr. 30.00

KIEPERT (L.). Grundriss der Differential- und Integralrechnung. 2ter Teil: Integralrechnung. 11te vermehrte Auflage. Hanover, Helwing, 1918. M. 17.00

——. Tabelle der wichtigsten Formeln aus der Integralrechnung. Hanover, Helwing, 1918. M. 1.20

KLEE (O.). Fermats letzter Satz. Berlin, Meyer und Müller, 1918.

KOWALEWSKI (G.). Einführung in die Infinitesimalrechnung. 3te Auflage. Leipzig, Teubner, 1918. M. 1.20

LIETZMANN (W.). Der pythagoreische Lehrsatz. Mit einem Ausblick auf das Fermatsche Problem. 2te Auflage. Leipzig, Teubner, 1918. 4 + 69 pp. M. 1.00

LINDOW (M.). Differentialrechnung. (Aus Natur und Geisteswelt, Nr. 387.) 2te Auflage. Leipzig, Teubner, 1918. 83 pp. Geb. M. 1.50

——. Integralrechnung mit Aufgabensammlung. (Aus Natur und Geisteswelt.) 2te Auflage. Leipzig, Teubner, 1918. Geb. M. 1.50

MEISSNER (O.). Wahrscheinlichkeitsrechnung. 2te Auflage. Leipzig, Teubner, 1918. M. 1.00

MENDELSSOHN (W.). Einführung in die Mathematik. (Aus Natur und Gesiteswelt, Nr. 503.) Leipzig, Teubner, 1918. 113 pp. Geb. M. 1.50

PANTON (A. W.). See BURNSIDE (W. S.).

PLANT (L. C.). See WEBBER (W. P.).

PRÜSS (A.). Ueber vollständig zerfallende algebraische Raumkurven und Flächen. (Diss.) Kiel, 1918.

ROSE (M.). Einleitung in die Funktionentheorie. Die komplexen Zahlen und ihre elementaren Funktionen. 2te verbesserte Auflage. Berlin, Göschen, 1918. M. 1.00

SCHROECKH (C.). Die Höchstzahl der reellen Züge einer Raumkurve $n$ter Ordnung in m Dimensionen. (Diss.) Kiel, 1917.

WEBBER (W. P.) and PLANT (L. C.). Introductory mathematical analysis. New York, Wiley, 1919. 13 + 304 pp. $2.00

WIELEITNER (H.). Der Begriff der Zahl in seiner logischen und historischen Entwicklung. (Mathematisch-physikalische Bibliothek.) 2te durchgesehene Auflage. Leipzig, Teubner, 1918. M. 1.00

ZIEHEN (T.). Das Verhältnis der Logik zur Mengenlehre. Berlin, Reuther und Reichard, 1917.

II. ELEMENTARY MATHEMATICS.

CRANTZ (P.). Arithmetik und Algebra zum Selbstunterricht. Leipzig, Teubner, 1918. 1ter Teil, 4 + 116 pp. 2ter Teil, 4 + 123 pp. M. 1.20 + 1.20

FISCHER (P. B.). Rechenaufgaben aus der Kriegszeit. Leipzig, Teubner, 1917. 48 pp. M. 1.20

HARTENSTEIN (F.). Fünfstellige logarithmische und trigonometrische Tafeln für den Schulgebrauch. 2te Auflage. Leipzig, Teubner, 1917.

HEATH (R. S.). Solid geometry including mensuration of surfaces and solids. 4th edition. London, Rivingtons, 1918. 123 pp. 4s.

JONES (F. T.). Problems and questions on algebra. Cleveland, University Supply and Book Company, 1919. 2 + 66 pp. $0.40

LÖTZBEYER (P.). Vierstellige Tafeln zum logarithmischen und Zahlenrechnen für Schule und Leben in neuer Anordnung. Leipzig, Teubner, 1918. 32 pp. M. 1.40

RAUSCHELBACH (H.). Divisionstafel enthaltend drei- oder vierziffrige Quotienten aller ein- bis dreiziffrigen Dividenden und aller zweiziffrigen Divisoren. Göttingen, 1918. M. 7.60

SCHMITZ (B.). Das neue Multiplikations-Verfahren. Leipzig, Adler, 1918. 13 pp. Geh. M. 2.00

——. "Triplex"; das neueste und vollkommenste Verfahren der amerikanischen Buchführung. Leipzig, Adler, 1918. 22 pp. + 4 plates. Geh. M. 3.00

SCHUSTER (A.). Mathematische Unterrichtsbriefe. Leipzig, Verlag Naturwissenschaften, 1918. M. 6.45

TIMERDING (H. E.). Der goldene Schnitt. (Mathematisch-physikalische Bibliothek, Band 32.) Leipzig, Teubner, 1918. M. 1.00

III. APPLIED MATHEMATICS.

ANUARIO del observatorio de Madrid para 1919. Madrid, Imprenta de la Casa Editorial Bailly-Baillière, 1918. 741 pp.

APPELL (P.) et DAUTHEVILLE (S.). Précis de mécanique rationnelle. Introduction à l'étude de la physique et de la mécanique appliquée. 2e édition. Paris, Gauthier-Villars, 1918. 8vo. 8 + 734 pp. Fr. 50.00

AQUINO (R. DE). The "newest" navigation altitude and azimuth tables. 2d stereotyped edition enlarged and improved. London, Potter, 1918. 8vo. 52 + 176 + 40 pp. 12s.

AUERBACH (F.). Die graphische Darstellung. 2te Auflage. Leipzig, Teubner, 1918. M. 1.20

BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik für Metallindustrieschulen, vorzugsweise für Maschinenbauschulen (Werkmeisterschulen), die Unterstufe der höheren Maschinenbauschulen und verwandten technischen Lehranstalten. Bearbeitet von S. Jakobi und A. Schlie. 4te Auflage. Leipzig, Teubner, 1917. 6 + 223 pp. M. 2.80

BLEICH (F.). Die Berechnung statisch unbestimmter Tragwerke nach der Methode des Viermomentensatzes. Berlin, 1918. M. 12.00

BÖRNSTEIN (R.). Die Lehre von der Wärme. 2te Auflage. Leipzig, Teubner, 1918. M. 1.50

BORTKIEWICZ (L. VON). Homogeneität und Stabilität in der Statistik. Upsala, Almquist und Wiksells, 1918.

DAUTHEVILLE (S.). See APPELL (P.).

DUNCAN (D. C.). See FERRY (E. S.).

FERRY (E. S.). A handbook of physics measurements. In collaboration with O. W. Silvey, G. W. Sherman and D. C. Duncan. 2 volumes. New York, Wiley, 1918. 8vo. 9 + 251 + 10 + 233 pp. $2.00 + $2.00

FÖPPL (A.). Vorlesungen über technische Mechanik. Band 3: Festigkeitslehre. 6te Auflage. Leipzig, 1918. M. 15.00

GRAY (A.). A treatise on gyrostatics and rotational motion. Theory and applications. London, Macmillan, 1918. Super royal 8vo. 2£ 2s.

HAUSSNER (R.). Darstellende Geometrie. 1ter Teil. 3te vermehrte und verbesserte Auflage. Berlin, 1918. M. 1.00

HENKEL (O.). Graphische Statik mit besonderer Berücksichtigung der Einflusslinien. 1ter Teil. Berlin, 1918. M. 1.00

HINRICKS (W.). Einführung in die geometrische Optik. Neudruck. Berlin, 1918. M. 1.00

HUNTER (J. DE G.). The earth's axis and triangulation. (Survey of India, Professional paper No. 16.) Dehra Dun, Office of the Trigonometrical Survey, 1918. 8 + 219 pp. + 6 charts. 5s. 4d.

JAKOBI (S.). See BARDEY (E.).

LÖTZBEYER (P.). Darstellende Geometrie des Geländes. Dresden, Ehlermann, 1918.

MANES (A.). Grundzüge des Versicherungswesens. 3te veränderte Auflage. Leipzig, 1918. M. 1.20

MEISSNER (O.). Anwendungen der Wahrscheinlichkeitsrechnung. Leipzig, Teubner, 1918. M. 1.00

MÜLLER (A.). Die Referenzflächen des Himmels und der Gestirne. Braunschweig, Vieweg, 1918. 162 pp. Geb. M. 6.60

MÜLLER (R.). Leitfäden für die Vorlesung über darstellende Geometrie. 3te Auflage. Braunschweig, Vieweg, 1917. 179 pp. M. 7.00

NORTON (A. P.). A star atlas and telescopic handbook (Epoch 1920) for students and amateurs. London and Edinburgh, Gall and Inglis, 1919. 25 pp. + 16 maps.

POINCARÉ (H.). Die neue Mechanik. 3te Auflage. Leipzig, 1918. M. 1.00

SCHLIE (A.). See BARDEY (E.).

SCHUDEISKY (A.). Leitfäden für den neuzeitlichen Linearzeichenunterricht. Handbuch für den Lehrer. Leipzig, Teubner, 1916. 8 + 81 pp. M. 4.80

SELME (L.). Principe de Carnot contre formule empirique de Clausius, essai sur le thermodynamique. Paris, Dunod et Pinat, 1917. 148 pp. Fr. 4.50

SHERMAN (G. W.). See FERRY (E. S.).

SILVEY (O. W.). See FERRY (E. S.).

VATER (R.). Praktische Thermodynamik. (Aus Natur und Geisteswelt, Nr. 596.) Leipzig, Teubner, 1918. 96 pp. Geb. M. 1.50

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. 2ter Band. Berlin, 1918. M. 12.00